1. Determine the analytical solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions

$$u(0, t) = u(1, t) = 0$$

and an initial distribution of

$$u(x, 0) = \sin(\pi x) + \sin(3\pi x)$$

2. For $u_t = \alpha u_{xx}$ the following discretization is proposed

$$\frac{3}{2}\frac{\Delta_t u_i^n}{\Delta t} - \frac{1}{2}\frac{\nabla_t u_i^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}\delta_x^2 u_i^{n+1}$$

Use the von Neumann stability analysis to obtain the stability condition for this technique.

3. Consider the model equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial u}{\partial x}$$

A numerical technique with the following formulation has been suggested for the FDE:

$$u_i^{n+\frac{1}{2}} = u_i^n + \frac{\alpha \Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right)$$

$$u_i^{n+1} = \frac{1}{2}\left(u_i^n + u_i^{n+\frac{1}{2}}\right) + \frac{\alpha \Delta t}{2\Delta x}\left(u_i^{n+\frac{1}{2}} - u_{i-1}^{n+\frac{1}{2}}\right)$$

Use the von Neumann stability analysis to obtain the stability condition for this technique. *Hint: Eliminate $n + \frac{1}{2}$ time level in second equation by the substitution of the first equation. The resulting equation will have time levels n and $n + 1$ only. Now apply the von Neumann stability analysis.*

4. For the following matrices

(a) Compute the eigenvalue.

(b) Compute the Gershgorin circles.

(c) Compare (1) and (2) according to the theorem.

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ 0 & \frac{1}{3} & 2 \end{bmatrix}$$